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Linear, Steady, Two-Dimensional Flows of Viscoelastic Liquids

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The general form of the equations describing a steady, two-dimensional flow of an incompressible liquid is reduced to a form containing only two parameters. The histories of stress and of deformation of a material element are written explicitly. A second-order, slow-flow approximation and a Maxwell type of constitutive equation are used to infer properties of the rheological behavior of viscoelastic liquids in such flows.

Engineering analysis of viscoelastic fluid mechanics requires the understanding of as large a variety of flow patterns as possible, so that flows of actual interest can be approximated satisfactorily with idealized flows which are rigorously tractable yet reasonably similar to the real ones. Furthermore, the rheological characterization of real fluids would be more reliable and complete if experiments could be made under conditions which are not restricted to only the classical viscometric flow patterns.

A substantial fraction of the literature concerning viscoelastic fluid mechanics is devoted to the analysis of viscometric flows (see, for example, reference *6*). A few different flow patterns have been considered in the literature: elongational flow (*4*), pure shear (*11*), and fourth-order flow (*2*).

If attention is limited to steady, two-dimensional flow patterns, only viscometric flow and pure shear have been considered; Astarita (*2*) has concisely considered a more complex flow pattern which results from a superposition of viscometric flow and pure shear.

In this paper, the general problem of steady, two-dimensional flow is analyzed in considerable detail. A complete description of the deformation history and of the stress history, in the sense of Oldroyd (*11*), of a material element is given explicitly. The two histories change continuously, at constant total shear rate, when the relative importance of the two independent components of the shear rate changes. Thus, if experiments are made for a real fluid, information can be obtained over a variety of different rheological histories. As suggested by Oldroyd (*12*), a sort of bookkeeping of such information may be the basis on which more general forms of the constitutive equation may be inferred.

BASIC KINEMATICS

The most general possible steady, two-dimensional flow pattern of an incompressible fluid can be described, in a locally Cartesian frame z'^i, by the following equations for the contravariant velocity vector v'^i:

$$\left.\begin{aligned} v'^1 &= \Gamma'_1 \; z'^2 + \gamma' z'^1 \\ v'^2 &= \Gamma'_2 \; z'^1 - \gamma' z'^2 \\ v'^3 &= 0 \end{aligned}\right\} \tag{1}$$

Only three parameters (Γ'_1, Γ'_2, γ') appear in Equations (1), because the requirement of continuity imposes the condition

$$\frac{\partial v'^1}{\partial z'^1} = -\frac{\partial v'^2}{\partial z'^2} \tag{2}$$

Inasmuch as the choice of the 1 and 2 indices is arbitrary, one may impose the condition

$$\Gamma'_1 \geqq \Gamma'_2 \tag{3}$$

by leaving the quantity γ' free to be either negative or positive.

Consider now a new Cartesian frame z^i which is obtained by rotating the z'^i frame by an angle α about the z^3 axis:

$$\left.\begin{aligned} z^1 &= z'^1 \; \cos\alpha - z'^2 \; \sin\alpha \\ z^2 &= z'^1 \; \sin\alpha + z'^2 \; \cos\alpha \\ z^3 &= z'^3 \end{aligned}\right\} \tag{4}$$

Two special cases (that is, viscometric flow and pure shear) are now easily recognized. If the following equality is fulfilled

$$\Gamma'_1 - \Gamma'_2 = 2\sqrt{\Gamma'^2 + \gamma'^2} \tag{5}$$

where

$$2\Gamma' = \Gamma'_1 + \Gamma'_2 \tag{6}$$

a choice of α such that

$$\sin\alpha = \frac{1}{\sqrt{2}}\sqrt{1 - \sqrt{\frac{\Gamma'^2}{\Gamma'^2 + \gamma'^2}}} \tag{7}$$

reduces Equations (1) to the form

$$\left.\begin{aligned} v^1 &= (\Gamma'_1 - \Gamma'_2) \; z^2 \\ v^2 &= v^3 = 0 \end{aligned}\right\} \tag{8}$$

Equations (8) are recognized as the equations describing a viscometric flow pattern; the shear rate $\Gamma'_1 - \Gamma'_2$ is, because of the choice leading to condition (3), implicitly positive.

The case of pure shear is obtained when

$$\Gamma'_1 = \Gamma'_2 = \Gamma' \tag{9}$$

In fact, if (9) is fulfilled, and α is again taken as given by Equation (7), Equations (1) are transformed into

$$\left.\begin{aligned} v^1 &= \sqrt{\gamma'^2 + \Gamma'^2} \; z^2 \\ v^2 &= \sqrt{\gamma'^2 + \Gamma'^2} \; z^1 \\ v^3 &= 0 \end{aligned}\right\} \tag{10}$$

Equations (10) are recognized as describing pure shear (*11*); again, the shear rate $\sqrt{\gamma'^2 + \Gamma'^2}$ is implicitly positive.

It is interesting to observe that if both Γ'_1 and Γ'_2 are zero, Equations (1) describe a two-dimensional elongational flow, such as discussed by Metzner (*7*). Thus, a *transformation* as discussed above [with α = 45 deg. as obtained from Equation (7) when $\Gamma' = 0$] reduces a two-dimensional flow pattern of elongational flow into a pure shear flow [condition (9) is of course satisfied when $\Gamma'_1 = \Gamma'_2 = 0$]. This equivalency does not seem to have been explicitly stated in the literature.

Let us now turn to the general case when neither (5) nor (9) is satisfied. If again the transformation called for in Equations (4) and (7) is considered, Equations (1) reduce to

$$\left.\begin{aligned} v^1 &= \Gamma_1 \; z^2 \\ v^2 &= \Gamma_2 \; z^1 \\ v^3 &= 0 \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} \Gamma_1 &= (\Gamma'_1 - \Gamma'_2)/2 + \sqrt{\Gamma'^2 + \gamma'^2} \\ \Gamma_2 &= (\Gamma'_2 - \Gamma'_1)/2 + \sqrt{\Gamma'^2 + \gamma'^2} \\ \Gamma_1 &\geqq \Gamma_2 \end{aligned}\right\} \tag{12}$$

Thus, the conclusion is drawn that the general flow pattern, Equations (1), can always be reduced to a simpler form, such as in Equations (11), which contain only two parameters. Of course, Equations (11) contain viscometric flow ($\Gamma_2 = 0$) and pure shear ($\Gamma_1 = \Gamma_2$) as special degenerate cases. For the sake of simplicity, the form (11) will be considered in the following, rather than the more complex yet equivalent form (1).

THE HISTORY OF DEFORMATION

Let τ be time measured backward from the instant of observation, and let ξ^i be a convected frame of reference in the sense of Oldroyd (*9, 10*), which coincides with the laboratory frame z^i at the instant of observation $\tau = 0$. Thus

$$\frac{dz^i}{d\tau} = -v^i \tag{13}$$

$$\tau = 0, \quad z^i = \xi^i \tag{14}$$

The set of differential equations (13) subject to boundary conditions (14) can be integrated to obtain the current coordinates of a material point $z^i(\xi^j, \tau)$ as a function of its coordinates at the instant of observation ξ^j. [The ξ^j are labels for each material point, which do not change with deformation; the choice of Equations (14) implies the use, as labels, of the coordinates at the instant of observation]:

$$\left.\begin{aligned} z^1 &= \xi^1 \cos h\left(\frac{\phi\tau}{2}\right) - \xi^2 \frac{\phi}{2\,\Gamma_2} \sin h\left(\frac{\phi\tau}{2}\right) \\ z^2 &= -\xi^1 \frac{\phi}{2\,\Gamma_1} \sin h\left(\frac{\phi\tau}{2}\right) + \xi^2 \cos h\left(\frac{\phi\tau}{2}\right) \\ z^3 &= \xi^3 \\ \phi &= 2\sqrt{\Gamma_1 \; \Gamma_2} \end{aligned}\right\} \tag{15}$$

The metric of the convected frame γ_{ij} can now be calculated directly:

$$\gamma_{ij} = \sum_k \frac{\partial z^k}{\partial \xi^i} \frac{\partial z^k}{\partial \xi^j} =$$

$$\begin{bmatrix} \dfrac{\Gamma_1 - \Gamma_2}{2\,\Gamma_1} + \dfrac{\Gamma}{\Gamma_1}\cos h\,(\phi\tau) & -\dfrac{2\Gamma}{\phi}\sin h\,(\phi\tau) & 0 \\ -\dfrac{2\Gamma}{\phi}\sin h\,(\phi\tau) & \dfrac{\Gamma_2 - \Gamma_1}{2\,\Gamma_2} + \dfrac{\Gamma}{\Gamma_2}\cos h(\phi\tau) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

where $2\Gamma = \Gamma_1 + \Gamma_2$.

Notice that $\gamma_{ij}(\tau)$ represents the history of deformation, in the sense that γ_{ij} is the Cauchy strain of the configuration at the current time τ as referred to the configuration at the instant of observation. In fact, the square of the distance among two material points at time τ is given by

$$(ds)^2|_\tau = \gamma_{ij}(\tau)\; d\xi^i d\xi^j = \gamma_{ij}(\tau)\; dz^i(0)\; dz^j(0) \tag{17}$$

In writing Equation (17), use has been made of the invariancy of $d\xi^i$ with deformation; at time $\tau = 0$, by definition, $d\xi^i = dz^i(0)$.

The components of the rate of strain tensor in the convected frame η_{ij} are obtained as

$$\eta_{ij} = -\frac{1}{2}\frac{\partial \gamma_{ij}}{\partial \tau} = \begin{bmatrix} -\frac{\phi}{2\Gamma_1}\sin h\,(\phi\tau) & \cos h\,(\phi\tau) & 0 \\ \cos h\,(\phi\tau) & -\frac{\phi}{2\Gamma_2}\sin h\,(\phi\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix}\Gamma \quad (18)$$

The components of the rate of strain tensor in the z^i frame e_{ij}, which of course could be calculated directly from Equations (11), are obtained as

$$e_{ij} = \Gamma \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (19)$$

It is important to point out that although the instantaneous rate of strain tensor e_{ij} only depends on the average shear rate Γ, the history of deformation, as given by either (16) or (18), also depends on the value of ϕ. Thus, even at constant total shear rate Γ, the history of deformation of a material element can be changed by changing the value of ϕ. In fact, the parameter $1/\phi$ is seen from Equation (18) to essentially represent a time scale for the history of deformation; the dependency of η_{ij} on elapsed time τ is through the group $\phi\tau$.

Let us define the invariant I_x of a generic tensor X_{ij} as

$$I_x = 2\, X_{ij}\, X^{ij} \quad (20)$$

The parameter ϕ is then recognized as

$$\phi = \sqrt{I_e - I_w} = 2\sqrt{\Gamma_1 \Gamma_2} = \sqrt{I_{e'} - I_{w'}} = 2\sqrt{\Gamma'_1\, \Gamma'_2 + \gamma'^2} \quad (21)$$

where w_{ij} is the vorticity tensor. The group $\sqrt{I_e - I_w}$ was introduced by Astarita (2) as the inverse of the appropriate time yardstick for the analysis of deformation histories. In the particular case of viscometric flow, $\phi = 0$, and the history of deformation degenerates into the well-known *(11)* polynomial form

$$(\eta_{ij})_{\phi=0} = \Gamma \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4\Gamma\tau & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (22)$$

The physical implications of the condition $\phi = 0$ and of the polynomial nature of η_{ij} have been discussed by Astarita (2).

The conjugate metric of the convected frame γ^{ij} is obtained very easily, because, as for all isochoric flows $\det(\gamma_{ij}) = 1$. Thus

$$\gamma^{ij} = \begin{bmatrix} \gamma_{22} & -\gamma_{12} & 0 \\ -\gamma_{12} & \gamma_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (23)$$

and, analogously, the contravariant form of η_{ij} is

$$\eta^{ij} = \frac{1}{2}\frac{\partial \gamma^{ij}}{\partial \tau} = \gamma^{ir}\,\gamma^{js}\,\eta_{rs} = \begin{bmatrix} -\eta_{22} & \eta_{12} & 0 \\ \eta_{12} & -\eta_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (24)$$

Higher order rate of strain tensors are easily calculated:

$$\overset{(2)}{\eta}_{ij} = -\frac{\partial \eta_{ij}}{\partial \tau} = \Gamma \begin{bmatrix} 2\,\Gamma_2 \cos h\,(\phi\tau) & -\phi \sin h\,(\phi\tau) & 0 \\ -\phi \sin h\,(\phi\tau) & 2\,\Gamma_1 \cos h\,(\phi\tau) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (25)$$

$$\overset{(N)}{\eta}_{ij} = (-1)^N \frac{1}{2}\frac{\partial^N \gamma^{ij}}{\partial \tau^N} = \phi^2\, \overset{(N-2)}{\eta}_{ij} \quad (26)$$

$$\overset{(2)}{\eta}{}^{ij} = \frac{\partial \eta^{ij}}{\partial \tau} = \begin{bmatrix} \overset{(2)}{\eta}_{22} & -\overset{(2)}{\eta}_{12} & 0 \\ -\overset{(2)}{\eta}_{12} & \overset{(2)}{\eta}_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (27)$$

$$\overset{(N)}{\eta}{}^{ij} = \frac{1}{2}\frac{\partial^N \gamma^{ij}}{\partial \tau^N} = \phi^2\, \overset{(N-2)}{\eta}{}^{ij} \quad (28)$$

From Equations (25) to (28), the Rivlin-Ericksen tensors $\overset{(N)}{A}_{ij}$ and the White tensors $\overset{(N)}{B}{}^{ij}$ (2, 15) are obtained directly

$$\overset{(N)}{A}_{ij} = \left(\overset{(N)}{\eta}_{ij}\right)_{\tau=0} \quad (29)$$

$$\overset{(N)}{B}{}^{ij} = \left(\overset{(N)}{\eta}{}^{ij}\right)_{\tau=0} \quad (30)$$

Needless to say, the B tensors are not obtained by raising indices on the A tensors. From Equations (26) and (28) the following well-known property of viscometric flow is immediately deduced (and seen to depend on the condition $\phi = 0$):

$$N > 2, \quad \overset{(N)}{A}_{ij} = \overset{(N)}{B}{}^{ij} = 0 \quad (31)$$

The equations given so far are a complete description of the kinematics of the flow pattern defined by Equations (11). In fact, all kinematic tensors are given explicitly as a function of time measured backward from the instant of observation; thus, the deformation of any material element is entirely described for any instant of time preceding observation.

BASIC DYNAMICS

If the flow pattern defined by Equations (11) is described in a new frame z''^i which differs from z^i only for the direction of the z^3 axis, Equations (11) would still be valid; thus, the stress tensor T''_{ij} would exactly coincide with T_{ij} (3). But, since

$$\left.\begin{aligned} T''_{13} &= -T_{13} \\ T''_{23} &= -T_{23} \end{aligned}\right\} \quad (32)$$

the conclusion is drawn that

$$T_{13} = T_{23} = 0 \quad (33)$$

Moreover, the stress tensor is symmetrical, and, if only incompressible fluids are considered, its trace is defined only within an arbitrary constant. Therefore, the stress tensor T_{ij} has only three independent components: T_{12}, $T_{11} - T_{22}$, and $T_{22} - T_{33}$. These components are uniquely determined by the values of Γ and ϕ, say

$$\left.\begin{aligned} T_{12} &= f_1(\Gamma, \phi) \\ T_{11} - T_{22} &= f_2(\Gamma, \phi) \\ T_{22} - T_{33} &= f_3(\Gamma, \phi) \end{aligned}\right\} \quad (34)$$

In the particular case of viscometric flow, $\phi = 0$, and the $f_i(\Gamma, 0)$ functions become the viscometric functions *(14)*.

By setting arbitrarily $T_{33} = 0$ (which is permissible in view of the incompressibility of the fluid), the stress tensor T_{ij} may be written as

$$T_{ij} = \begin{bmatrix} f_2 + f_3 & f_1 & 0 \\ f_1 & f_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (35)$$

Of course, in order to obtain an analytic formulation of the f_i's, a constitutive equation has to be assumed. Yet, the history of stress, that is, the $\tau_{ij}(\tau)$ function where τ_{ij} is the stress component in the ξ^i frame, can be obtained with-out specifying the form of the f_i by following a procedure originally suggested by Oldroyd (*11*).

THE HISTORY OF STRESS

It is our purpose to obtain the $\tau_{ij}(\xi^k, \tau)$ function, which describes the history of stress at the material point ξ^k. In view of Equations (14), we know that

$$\tau_{ij}(\xi^k, 0) = T_{ij} \qquad (36)$$

Let us now consider a new convected frame ξ'^i defined by

$$\left.\begin{aligned} \xi'^1 &= \xi^1 \cos h\left(\frac{\phi t}{2}\right) - \xi^2 \frac{\phi}{2\,\Gamma_2} \sin h\left(\frac{\phi t}{2}\right) \\ \xi'^2 &= -\xi^1 \frac{\phi}{2\,\Gamma_1} \sin h\left(\frac{\phi t}{2}\right) + \xi^2 \cos h\left(\frac{\phi t}{2}\right) \\ \xi'^3 &= \xi^3 \end{aligned}\right\} \qquad (37)$$

where t is an arbitrary number. By considering Equations (15), the ξ'^i frame is seen to coincide with the laboratory frame z^i at time $\tau = t$:

$$\tau = t, \quad \xi'^i = z^i \qquad (38)$$

Thus, obviously

$$\tau'^{ij}(\xi'^k, \tau = t) = T_{ij} \qquad (39)$$

But, by the known rules of transformation of tensors

$$\tau_{ij}(\xi^k, \tau = t) = \frac{\partial \xi'^r}{\partial \xi^i}\,\frac{\partial \xi'^s}{\partial \xi^j}\,\tau'_{rs}(\xi'^k, \tau = t) \qquad (40)$$

Considering Equation (39), and setting $t = \tau$ in Equations (37), one gets

$$\tau_{ij}(\xi^k, \tau) = a_i^r\ a_j^s\ T_{rs} \qquad (41)$$

where

$$a_j^i = \begin{bmatrix} \cosh\dfrac{\phi\tau}{2} & -\dfrac{\phi}{2\,\Gamma_2}\sinh\dfrac{\phi\tau}{2} & 0 \\ -\dfrac{\phi}{2\,\Gamma_1}\sinh\dfrac{\phi\tau}{2} & \cosh\dfrac{\phi\tau}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (42)$$

Equation (41) gives the required history of stress.* Inasmuch as $\tau_{13} = \tau_{23} = \tau_{33} = 0$, there are only three independent components of τ_{ij}, namely

$$\tau_{11} = \left(\frac{f_2 + f_3}{2} - \frac{f_3}{2}\,\frac{\phi^2}{4\,\Gamma_1^2}\right) + \left(\frac{f_2 + f_3}{2} + \frac{f_3}{2}\,\frac{\phi^2}{4\,\Gamma_1^2}\right)\cosh(\phi\tau) - f_1\frac{\phi}{2\,\Gamma_1}\sinh(\phi\tau) \qquad (43)$$

$$\tau_{12} = f_1\cosh(\phi\tau) - \left(\frac{f_2 + f_3}{2}\,\frac{\phi}{2\,\Gamma_2} + \frac{f_3}{2}\,\frac{\phi}{2\,\Gamma_1}\right)\sinh(\phi\tau) \qquad (44)$$

$$\tau_{22} = \left(\frac{f_3}{2} - \frac{f_2 + f_3}{2}\,\frac{\phi^2}{4\,\Gamma_2^2}\right) + \left(\frac{f_3}{2} + \frac{f_2 + f_3}{2}\,\frac{\phi^2}{4\,\Gamma_2^2}\right)\cosh(\phi\tau) - f_1\frac{\phi}{2\,\Gamma_2}\sinh(\phi\tau) \qquad (45)$$

The contravariant form is, of course, immediately obtainable because the metric of the convected frame is known.

THE CASE WHERE $\Gamma_2 < 0$

By considering Equations (3) and (12), the parameter Γ_1 is seen to be essentially positive. Nonetheless, Γ_2 may be negative if the original values of Γ'_1 and Γ'_2 are of opposite sign. This is a physically possible situation which needs to be briefly discussed.

If $\Gamma_2 < 0$, the value of ϕ is imaginary. Nonetheless, all the equations given so far are still valid. By defining a new parameter Φ as:

$$\Phi = i\phi = 2\sqrt{-\Gamma_1\ \Gamma_2} \qquad (46)$$

the following substitutions result in the elimination of the imaginary forms in all the equations above:

$$\left.\begin{aligned} \phi^2 &= -\Phi^2 \\ \cosh(\phi\tau) &= \cos(\Phi\tau) \\ \phi\sinh(\phi\tau) &= \Phi\sin(\Phi\tau) \\ \frac{\sinh(\phi\tau)}{\phi} &= \frac{\sin(\Phi\tau)}{\Phi} \end{aligned}\right\} \qquad (47)$$

Physically, the case $\Gamma_2 < 0$ corresponds to a periodic motion with a period $1/\Phi$.

SLOW-FLOW APPROXIMATION

A slow-flow approximation in the sense of Coleman and Noll (3), carried out to the second order, yields the constitutive equation:

$$\frac{1}{\mu}T_{ij} = 2\ e_{ij} + 4\ \alpha_o\ \theta\ e_{ik}\ e_j{}^k - \beta_o\ \theta\ \overset{(2)}{A}_{ij} \qquad (48)$$

or, in terms of contravariant tensors

$$\frac{1}{\mu}T^{ij} = 2\ e^{ij} + 4\ \alpha^o\ \theta\ e^{ik}\ e_k{}^j + \beta^o\ \theta . \overset{(2)}{B}{}^{ij} \qquad (49)$$

where μ is the zero-shear viscometric viscosity, α_o and α^o are dimensionless cross-viscosity coefficients of the Reiner-Rivlin type (*13*), β_o and β^o are dimensionless coefficients of the linear viscoelastic type, and θ is the natural time of the liquid (*1*).

Upon substitution of the appropriate kinematic variables, Equations (48) and (49) give

$$\left.\begin{aligned} T_{12} &= 2\,\mu\,\Gamma \\ \frac{T_{11} - T_{22}}{T_{12}} &= \beta_o\ \theta\ (\Gamma_1 - \Gamma_2) \\ \frac{T_{22} - T_{33}}{T_{12}} &= \theta\ ([\alpha_o - \beta_o]\ \Gamma_1 + \alpha_o\ \Gamma_2) \end{aligned}\right\} \qquad (50)$$

$$\left.\begin{aligned} T^{12} &= 2\,\mu\,\Gamma \\ \frac{T^{11} - T^{22}}{T^{12}} &= \beta^o\ \theta\ (\Gamma_1 - \Gamma_2) \\ \frac{T^{22} - T^{33}}{T^{12}} &= \theta\ (\alpha^o\ \Gamma_1 + [\alpha^o + \beta^o]\ \Gamma_2) \end{aligned}\right\} \qquad (51)$$

Equations (50) and (51) are indistinguishable. In fact, the first two equations of each set are equivalent; the third equation is only apparently different. Suppose that a plot of $(T_{22} - T_{33})/T_{12}$ is made vs. Γ_2 at constant Γ_1. A straight line would be expected from both Equations (50) and (51), the slope and intercept being related to each other by

$$\text{intercept} = \left(\text{slope} - \frac{T_{11} - T_{22}}{T_{12}(\Gamma_1 - \Gamma_2)}\right)\Gamma_1 \qquad (52)$$

The same is true for higher order approximations; the covariant and contravariant forms of the constitutive equation are equivalent if the approximation is carried out correctly (*2*).

In the particular case when $\Gamma_1 = \Gamma_2$, Equations (50) and (51) of course predict that $T_{11} = T_{22}$. In the case of viscometric flow, Equations (50) and (51) degenerate

* In this particular case, τ_{ij} does not depend on the ξ^k; that is, the stress field is uniform.

into the well-known, second-order approximations of the viscometric functions (*14*), which again are equivalent in the covariant and contravariant forms.

MAXWELL FLUID APPROXIMATION

A number of viscoelastic constitutive equations have been discussed in the literature, and the stresses predicted by each one of these can easily be calculated for the flow pattern discussed here by simply substituting the appropriate terms. Discussion here will be limited to the so-called Maxwell equation:

$$T^{ij} = 2\ \mu\ e^{ij} - \theta \frac{\mathfrak{d}}{\mathfrak{d}t} T^{ij} \tag{53}$$

Conclusions similar to those obtained below would be deduced by any constitutive equation which implies an essentially exponential rate of stress relaxation, as discussed by Astarita (*2*).

In Equation (53), μ and θ are the zero-shear viscosity and the natural time, and $\mathfrak{d}/\mathfrak{d}t$ is the convected derivative introduced by Oldroyd (*10*)

$$\frac{\mathfrak{d}}{\mathfrak{d}t} T^{ij} = \frac{\partial T^{ij}}{\partial t} + v^k\ T^{ij}{}_{,k} - v^i{}_{,k}\ T^{kj} - v^j{}_{,k}\ T^{ik} \tag{54}$$

Equation (53) has the advantage of being at the same time very simple in character, yet capable of predicting at least qualitatively the behavior of real fluids under rapid flow conditions (*1*, *8*). A formulation of (53) in terms of covariant components is not equivalent; the contravariant form is chosen here because it predicts the approximately correct normal stress pattern for real fluids in viscometric flow (*5*).

For the flow pattern of interest, the derivative $\mathfrak{d}T^{ij}/\mathfrak{d}t$ is

$$-\frac{\mathfrak{d}T^{ij}}{\mathfrak{d}t} = \begin{bmatrix} 2\,\Gamma_1\ T^{12} & \Gamma_1\ T^{22} + \Gamma_2\ T^{11} & 0 \\ \Gamma_1\ T^{22} + \Gamma_2\ T^{11} & 2\,\Gamma_2\ T^{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

Substitution into Equation (53) yields

$$\left.\begin{aligned} T^{12} &= \frac{2\,\mu\,\Gamma}{1-\theta^2\,\phi^2} \\ T^{11} - T^{22} &= \frac{4\,\mu\,\theta\,\Gamma}{1-\theta^2\,\phi^2}\,(\Gamma_1 - \Gamma_2) \\ T^{22} - T^{33} &= \frac{4\,\mu\,\theta\,\Gamma_2\Gamma}{1-\theta^2\,\phi^2} \end{aligned}\right\} \tag{56}$$

Many interesting considerations can be made concerning Equations (56). First of all, it should be noted that all the stress components are predicted to become infinity when

$$\phi \to 1/\theta \tag{57}$$

Yet, if the value of ϕ is below the limit of Equation (57), the stress components are finite whatever the value of Γ. The physical interpretation of this point has been discussed by Astarita (*2*).

It is clear that in an actual flow pattern the internal stresses cannot become infinite; thus, when the flow pattern is such that condition (57) is approached, a real fluid will relieve the stresses through fracture, that is, by losing geometrical continuity. A mechanism of this type may be responsible for the well-known phenomenon of fracture observed during extrusion of polymer melts.

Another important conclusion is obtained by considering the apparent viscosity, that is, the ratio of the tangential stress T^{12} to the shear rate 2Γ:

$$\frac{T^{12}}{2\Gamma} = \frac{\mu}{1-\theta^2\ \phi^2} \tag{58}$$

Equation (58) shows that the apparent viscosity is not uniquely determined by the shear rate. At constant Γ the apparent viscosity increases with increasing ϕ; that is, it depends on the form of the history of deformation as given by Equations (16) and (18). The lowest value of the apparent viscosity is the viscometric viscosity, which for the Maxwell fluid coincides with the zero-shear viscosity; the highest value of the apparent viscosity is obtained in pure shear when $\phi = \Gamma$.

Finally, Equations (56) show that while in viscometric flow the normal stresses are even functions of the shear rate, the same is not true when $\phi \neq 0$; in the limit of pure shear, the difference $T^{22} - T^{33}$ is an odd function of the shear rate. The tangential stress is always an odd function of Γ.

NOTATION

- $\overset{(N)}{A}_{ij}$ = N^{th} Rivlin-Ericksen tensor
- $a_j^{\,i}$ = coordinate transformation matrix
- $\overset{(N)}{B}{}^{ij}$ = N^{th} White tensor
- ds = distance among two neighboring material points
- e_{ij} = rate of strain tensor's components in the z^i system
- f_1, f_2, f_3 = stress functions
- I_x = invariant of tensor X_{ij}, see Equation (20)
- N = integer
- T_{ij}, T''_{ij} = stress tensors
- t = time
- v^i, v'^i = velocity vectors
- z^i, z'^i, z''^i = Cartesian laboratory coordinate systems
- $\mathfrak{d}/\mathfrak{d}t$ = convected time derivative operator

Greek Letters

- α = angle of rotation
- α_o, α^o, β_o, β^o = rheological coefficients
- Γ, Γ', γ' = shear rates
- γ_{ij} = metric of the convected system
- η_{ij} = convected rate of strain tensor
- $\overset{(N)}{\eta}_{ij}$ = N^{th} rate of strain tensor
- θ = natural time
- μ = zero-shear viscosity
- ξ^i, ξ'^i = convected coordinate systems
- τ = time lag
- τ_{ij}, τ'_{ij} = convected stress tensors
- ϕ = characteristic reciprocal time, see Equation (15)
- Φ = $i\phi$

The summation convention and comma notation are used throughout.

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